THE DEVELOPMENT OF A DUAL ECONOMY[1]

1. Two decades of rapid progress in the theory of economic growth and development [2] have left some curious gaps in the received doctrine. On the one hand there is a large and steadily increasing battery of theories or "models" alleged to apply to advanced economies. Of these the most familiar and time-tested is the Harrod–Domar theory of growth.[3] Three recent entries to the field which must be noted are the Duesenberry–Smithies model [4] of cycles and growth, the Tobin–Solow "neo-classical" growth model [5] and the Kaldor model of growth.[6] Of course, this is only part of the array of theoretical devices now available to students of growth in advanced economies.[7] Whatever the differences among these growth theories, they share one distinctive feature: The analysis throughout is on the highest level of aggregation. There is assumed to be a single commodity and a single producing sector. On the other hand, there is a smaller number of theories of development for backward economies. Most important among these contributions, from a theoretical point of view, is the recent work of Leibenstein,[8] carrying forward the program of creating an economic-demographic theory of development.[9] As in the theory of growth for an advanced economy, the analysis is carried through on a level of aggregation which permits only one output and a single production relation. The differences between the theory of growth (for advanced economies) and the theory of

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[2] Dating the beginning of serious work in this field with R. F. Harrod's "An Essay in Dynamic Theory," ECONOMIC JOURNAL, Vol. 49, 1939, pp. 14–33.

[3] Harrod, *ibid.*, and *Towards a Dynamic Economics* (London: Macmillan, 1948); E. D. Domar, "Capital Expansion, Rate of Growth, and Employment," *Econometrica*, Vol. 14, 1946, pp. 137–47.

[4] J. S. Duesenberry, *Business Cycles and Economic Growth* (New York: McGraw-Hill, 1958); A. Smithies, "Economic Fluctuations and Growth," *Econometrica*, Vol. 25, 1957, pp. 1–52.

[5] R. M. Solow, "A Contribution to the Theory of Economic Growth," *Quarterly Journal of Economics*, Vol. 70, 1956, pp. 65–94; J. Tobin, "A Dynamic Aggregative Model," *Journal of Political Economy*, Vol. 63, 1955, pp. 103–15.

[6] N. Kaldor, "A Model of Economic Growth," ECONOMIC JOURNAL, Vol, 67, 1957, pp. 591–624; reprinted in *Essays in Economic Stability and Growth* (Duckworth, 1960); reference should also be made to: "Capital Accumulation and Economic Growth" to appear in the forthcoming proceedings of the International Economic Association's Corfu conference on capital theory, held in September 1958.

[7] A helpful review and exposition of dynamic theory is: W. Baumol, *Economic Dynamics*, 2nd edition (London: Macmillan, 1959).

[8] See especially H. Leibenstein, *Economic Backwardness and Economic Growth* (New York: Wiley, 1957).

[9] H. Leibenstein, *A Theory of Economic-Demographic Development* (Princeton: Princeton University Press, 1954).

development (for backward economies) are so great that one might be tempted to conclude that there is little in common between them. In the theory of development emphasis is laid on the balance between capital accumulation and the growth of population, each adjusting to the other. In the theory of growth the balance between investment and saving is all-important and the growth of population is treated as constant or shunted aside as a qualification to the main argument.[1] However great the differences between theories of growth and development appear to be, it is hard to escape the conclusion that each of the theories is true, sometimes. No simple model seems to be true all of the time or even very often. In particular, limitation of the analysis to situations in which there is effectively one producing sector rules out much of what is interesting about growth and development, at least if the empirical and institutional literature is any guide.[2] A few examples of " special situations " or " unsolved problems " created by concentration on a single output or a single production relation are: balance between industries in economic growth, imbalance between advanced and backward countries in international trade and the development of a dual economy, that is, of an economy with an advanced or modern sector and a backward sector as well.[3] These problems remain virtually untouched in the theoretical literature.[4] The purpose of this paper is to begin a frontal attack on the gap between theories of growth and theories of development by presenting a theory of development of a dual economy. The theory to be presented here is not intended to be yet another candidate for *the* universal theory of economic growth and development,[5] but only a theory which is applicable to a few well-defined and empirically significant historical situations. Whatever loss of generality is suffered in striving for something less than a

[1] For an exception to this rule see the interesting but brief theoretical treatment of population growth in Solow, *op. cit.*, pp. 90–1.

[2] See especially: W. A. Lewis, *The Theory of Economic Growth* (London: Allen and Unwin, 1955); and Part II of G. M. Meier and R. E. Baldwin, *Economic Development* (New York: Wiley, 1957); for a survey of the historical and institutional literature on development. For references to a dual economy see footnote 3, below.

[3] The source of the notion of a dual economy is: J. H. Boeke, *Economics and Economic Policy of Dual Societies* (Haarlem: Tjeenk Willnik, 1953) (earlier edition in two vols., 1942, 1946), which deal with the case of Indonesia; following Boeke's orientation is the important study by R. Firth, *Malay Fisherman* (London: Paul, Trench, Trubner, 1946), which discusses the traditional sector of the Malayan economy. For a critical view of Boeke's thesis and his policy recommendations, see B. Higgins, " The ' Dualistic Theory ' of Underdeveloped Areas," *Economic Development and Cultural Change*, Vol. 4, 1956, pp. 99–115. The notion of a " dual economy " has been employed by Chenery, Clark, Mandelbaum, Rosenstein-Rodan and Singer, among others.

[4] But see G. O. Gutman, "A Note on Economic Development with Subsistence Agriculture," *Oxford Economic Papers*, Vol. 9, 1957, pp. 323–9, with mathematical appendix by J. Black. Black uses the well-known two-country trade model of H. Johnson: " Increasing Productivity, Income–Price Trends and the Trade Balance," ECONOMIC JOURNAL, Vol. 64, 1954, pp. 462–85; reprinted in *International Trade and Economic Growth* (London: Allen and Unwin, 1958), pp. 94–119.

[5] For recent assaults on the summit see Joan Robinson, *The Accumulation of Capital* (London: Macmillan, 1958); D. G. Champernowne, " Capital Accumulation and Full Employment," ECONOMIC JOURNAL, Vol. 68, 1958, pp. 218–44; and the references of footnotes 4–6 on p. 309.

single theory of growth must be balanced against the gain in specific implications for those situations to which the theory of development of a dual economy does apply.[1]

Very briefly, the situation envisaged in the theory of a dual economy is this: The economic system may be divided into two sectors—the advanced or modern sector, which we will call, somewhat inaccurately, the manufacturing sector, and the backward or traditional sector, which may be suggestively denoted agriculture.[2] Productive activity in each sector may be characterised by a function relating output to each of the factors of production—land, labour and capital. The special character of the theory of development of a dual economy is a certain asymmetry in the productive relations. If the two production functions were essentially symmetric, that is, if each function included all three productive factors, the resulting model would be suited to the problems of industrial balance in an advanced economy or to dynamic problems in the theory of international trade.[3] In the theory of a dual economy the output of the traditional or agricultural sector is a function of land and labour alone; there is no capital accumulation, except where investment takes the form of land reclamation. For the purposes of the present analysis it seems best to assume that all potentially arable land is under cultivation, so that land is fixed in supply. Land does not appear as a factor of production in the manufacturing sector; the level of manufacturing output is a function of capital and labour alone. Agricultural activity is characterised by diminishing returns to scale. Although there are many ways to account for diminishing returns—*e.g.*, declining quality of the land as more and more is put under cultivation as in Ricardo's extensive margin [4]—the initial assumption that land is fixed in supply (and of one uniform quality, arable) implies that the diminishing returns encountered in agricultural production arise at the intensive margin of the Ricardian scheme. The interpretation of diminishing returns and increasing returns as the effect of a fixed factor has been stressed by Kaldor.[5] In manufacturing, expansion of productive activity proceeds with constant returns to scale. This appears to be a reasonable assumption, at least on the basis of evidence from the manufacturing industries of advanced economies.[6] A second feature of the production functions for agriculture and

[1] The situation envisaged is that of Japan, as well as the areas of South-East Asia studied by Boeke and Firth. The idea of a dual economy evidently has considerably broader empirical relevance.

[2] In the primary-producing countries of South-East Asia the advanced sector is plantation agriculture, mining and extraction of petroleum. The traditional sector is peasant agriculture and fishing. In Japan the traditional sector includes agriculture, small manufacturing and most construction; the advanced sector should be identified with Japanese heavy industry.

[3] That is, to the problems treated by Johnson, *op. cit.*

[4] D. Ricardo, *Principles* (London: Dent, 1911 (first edition 1817)), p. 35.

[5] N. Kaldor, "The Equilibrium of the Firm," ECONOMIC JOURNAL, Vol. 44, 1934, pp. 60–76.

[6] Reference may be made to the theory of the L-shaped cost curve: P. Wiles, *Price, Cost and Output* (Oxford: Blackwell, 1956), Chapter 12 and appendix, pp. 202–51; and J. S. Bain, *Barriers to New Competition* (Cambridge: Harvard, 1956), Chapter 3, pp. 53–113.

industry is that each function will shift over time so that a given bundle of factors will generate a higher level at one date than at an earlier date. The autonomous shifts in the production function correspond to technological changes. Although it is possible to devise purely economic explanations for some observed changes in technology,[1] we will assume that the changes take place at some more or less constant rate and that all changes are neutral.[2]

The assumption that technical changes are neutral is by now customary in the analysis of growth; whatever the theoretical importance of biased technical progress, it is hard to see how one may distinguish empirically between shifts of a production function, movements along it and shifts in the relative factor productivities, simultaneously.[3]

The characteristic feature of the theory of development of a dual economy is the asymmetry between production relations in the industrial and agricultural sectors. The remaining elements of the theory may be enumerated briefly: Population growth depends on the supply of food *per capita* and the force of mortality; the force of mortality is assumed given—it may be altered only by an alteration in medical technique. The birth rate depends on the supply of food *per capita*; however, it may attain a physiological maximum, or a maximum given by a particular social situation and by the state of medical knowledge, provided that the supply of food is sufficient. If the food supply is more than sufficient there exists an agricultural surplus, and labour may be freed from the land for employment in manufacturing. Labour is divided between the two sectors in a straightforward manner: if there is no agricultural surplus, all labour remains on the land; if an agricultural surplus can be generated, a labour force available for employment in manufacturing grows at a rate which is equal to the rate of growth of the agricultural surplus. Of course, no manufacturing production is possible without some initial capital stock, however small the initial bundle may be. Once the initial injection of capital is made, capital formation proceeds at a pace determined by the growth of the manufacturing labour force and by the terms of trade between the two sectors. It would not be correct to assume, as one might more readily assume in an advanced economy, that the real wage-rate is the same in the two sectors. In the first place only labourers in the advanced urbanised sector can be assumed to respond to wage differentials between employment opportunities in agriculture and industry. Since the course of development will result, with few lapses, in a steady

[1] See, for example, the interesting work of Zvi Griliches: "Hybrid Corn: An Exploration in the Economics of Technological Change," *Econometrica*, Vol. 25, 1957, pp. 501–22.

[2] A technological change is "neutral" provided that for a given bundle of factors the marginal rate of substitution between factors with output held constant is the same before and after the change.

[3] There is a considerable literature on this problem, mainly unpublished. The key reference is R. M. Solow, "Technical Change and the Aggregate Production Function," *Review of Economics and Statistics*, Vol. 39, 1957, pp. 312–20, where shifts of a production function and movements along it are separately identified. It is an open question whether it is possible to test the assumption of neutrality of technical change which underlies Solow's work.

migration of labour from the backward agricultural sector to the modern sector, some differential between wages in agriculture and wages in manufacturing may be expected to persist. We will assume that this differential is proportional to the manufacturing wage-rate and is stable in the long run. This differential determines the terms of trade between manufacturing and agricultural sectors, and thereby the rate of investment in the modern or advanced sector. Under this interpretation of the model it must be assumed either that the economy is closed to trade or that trade is in balance, not only overall but also in the goods of each sector. Such a balance could occur, for example, whenever there are no imports or exports of food and trade is in balance. Then manufactured goods would be traded for manufactured goods and the production function of the advanced sector would consist of a relationship between factors employed in manufacturing and domestic utilisation of manufactured goods, including imports and excluding exports. The assumption that the economy is essentially closed to trade follows the practice of most recent contributions to the theory of growth and development.[1] An alternative interpretation of the model is that the terms of trade between agriculture and industry in domestic commerce are determined by the rate of substitution of manufactured goods for agricultural goods in international trade. Then the wage differential between agriculture and industry would be determined by the terms of trade. In what follows we confine the analysis to a closed economy or an economy in which trade is in balance for goods of both sectors.

2. To begin the analysis we consider an economic system in which no development of manufacturing activity has taken place; all productive activity is concentrated in the traditional or backward sector. Then, if P is the total population and Y is agricultural output with L the fixed quantity of land available to the society, a simple version of the production function in agriculture, characterised by constant returns with all factors variable is given by the Cobb–Douglas production function:[2]

$$Y = e^{\alpha t} L^{\beta} P^{1-\beta}$$

where $e^{\alpha t}$ represents the shift factor corresponding to technological progress. Changes in techniques are assumed to take place at a constant rate, α. The constant β represents the elasticity of output with respect to an increase in the supply of land; as is well known, it also corresponds to the share of landlords in the product of the traditional sector; if the supply of land is

[1] See the references in footnotes 4–6, p. 309.

[2] The basic reference to the work of Douglas is " Are There Laws of Production," *American Economic Review*, Vol. 38, 1948, pp. 1–42; many further references are listed there. The so-called Cobb–Douglas form was apparently introduced into the literature by Wicksell in his notable article " A Mathematical Analysis of Dr. Akerman's Problem," *Ekonomisk Tidskrift*, Vol. 25, 1923, pp. 157–80; translated into English by E. Classen, and reprinted as part of an appendix to Wicksell's *Lectures on Political Economy*, Vol. I (London: Routledge and Kegan Paul, 1934). For a recent discussion of " Akerman's problem " reference must be made to the contribution of R. M. Solow to the Corfu conference on capital theory.

fixed, as in the model described here, this share takes the form of rent, defined as the unimputed residual remaining after the share of labour in the product, $1 - \beta$, has been paid to the agricultural labour force. Since the supply of land is assumed fixed, it is possible to choose the origin for measuring the passage of time so that the production function may be rewritten in the simpler form:

$$Y = e^{\alpha t} P^{1-\beta}$$

Dividing both sides by the agricultural population, we have:

$$y = \frac{Y}{P} = e^{\alpha t} P^{-\beta}$$

where y is agricultural output per man. Differentiating with respect to time and dividing through by output per man, this production function takes the form:

$$\frac{\dot{y}}{y} = \alpha - \beta \frac{\dot{P}}{P}$$

where α is the rate of technical progress and β is defined as before. This equation characterises the agricultural production function completely for the analysis to follow.

The second characteristic feature of the agricultural system is described by the function which governs the growth of population. It is assumed that if there is literally no agricultural production the reproduction rate falls to zero and the force of mortality is constant at the level δ. Secondly, it is assumed that the rate of gross reproduction is an increasing function of agricultural output per head up to some physiological maximum, say $\epsilon + \delta$. It is simplest to assume that the rate of increase in the gross reproduction rate is constant as *per capita* income increases, so that the population theory may be written in the form:

$$\frac{\dot{P}}{P} = \min \begin{cases} \gamma y - \delta \\ \epsilon \end{cases}$$

where γ is the rate of increase in the gross reproduction rate with respect to an increase in the output of food per man. The net reproduction rate $\dot{P}/P$ is the gross reproduction rate less the force of mortality, where the gross reproduction rate is the minimum of the two rates determined by the physiologically maximum rate of reproduction and the rate determined by the output of food per head.

For the first phase of development, in which reproduction is below its physical maximum, the population function and the production function may be combined to give a single differential equation in output per head alone:

$$\frac{\dot{y}}{y} = \alpha - \beta(\gamma y - \delta) = \alpha + \beta\delta - \beta\gamma y$$

Now, multiplying by y, we obtain the fundamental differential equation for the theory of development of the agricultural sector:

$$\dot{y} = (\alpha + \beta\delta)y - \beta\gamma y^2$$

To characterise the possible modes of development for an economic system which satisfies the conditions defining the production function and the theory of population growth, we will analyse the solutions of this equation. It should be noted first that there are two stationary solutions of the equation, that is, values of *per capita* income, which once established will maintain themselves. All such stationary solutions may be found by setting the rate of change in *per capita* income to zero, as follows:

$$(\alpha + \beta\delta)y - \beta\gamma y^2 = 0$$

This equation has two roots, $y_1 = 0$, and $y_2 = (\alpha + \beta\delta)/\beta\gamma$, which is necessarily positive. If the level of agricultural output reaches zero the rate of population growth falls to the negative rate given by the force of mortality, $-\delta$, and population dies off exponentially. No further attention need be devoted to this case; we confine further analysis to the stationary solution y_2. We have already characterised y_2 as a situation in which *per capita* output of food remains constant. The rate of population growth must, however, be positive. This may be seen by inserting the given value for y_2 into the equation for determining the rate of growth of population:

$$\frac{\dot{P}}{P} = \gamma[(\alpha + \beta\delta)/\beta\gamma] - \delta = \frac{\alpha}{\beta} > 0$$

so that population and the food supply grow at the same positive rate with no progress, that is, no increase in the output of food *per capita*. The situation in which output per head remains stagnant and population growth is positive is precisely that envisioned in Leibenstein's low-level equilibrium trap.[1] Although there are important differences between the model described here and that of Leibenstein, the existence of a kind of " trap " level of income suggests the problem: Under what conditions is the low-level equilibrium trap stable? In formal terms: Under what conditions is y_2 a stable solution of the fundamental differential equation? The answer is simple: If a low-level equilibrium trap would be stable, if it existed, then it must exist. In this case existence of stationary equilibrium and its stability are equivalent.

To complete the analysis of a pure traditional or agricultural system, let us denote by y^+ the minimum level of income at which $\dot{P}/P$ attains its physiological maximum. Then, using the model for population growth, we may solve for y^+ as follows:

$$\frac{\dot{P}}{P} = \gamma y^+ - \delta = \epsilon$$

[1] Leibenstein, *Economic Backwardness and Economic Growth*, Chapter 3, and the references listed there, p. 15.

Hence:

$$y^{+} = (\epsilon + \delta)/\gamma$$

is the value of production of food per head at which population growth attains its maximum rate. The two cases into which solutions to the fundamental differential equation may be divided are distinguished as follows: If the critical level of income, y^{+}, is below y_2, the stationary level assuming that population growth has not attained its maximum, y_2 will not be attained, since it would require a rate of growth in population higher than the physiological maximum. In the event that y_2 exceeds y^{+} the growth path will be given instead by the expression:

$$\frac{\dot{y}}{y} = \alpha - \beta\epsilon$$

Multiplying both sides by y we have:

$$\dot{y} = (\alpha - \beta\epsilon)y$$

which has the general solution:

$$y(t) = e^{(\alpha - \beta\epsilon)t}\, y(0)$$

where the rate of growth in *per capita* income, $\alpha - \beta\epsilon > 0$, will be attained from any positive initial level of output. Note that if this rate of growth is positive, $y_2 > y^{+}$ and no stationary level of equilibrium (except $y_1 = 0$) exists. The second case is that in which $y_2 < y^{+}$. In this case the stationary level y_2 exists and is stable in the large, that is y_2 will be approached for any positive initial level of agricultural output. If the initial level of output is sufficiently high that population growth attains its maximum the rate of growth in output per head, $\alpha - \beta\epsilon$, is negative and output declines at this rate to the level y^{+}. From this point the original fundamental differential equation describes the further decline of *per capita* production of food to its equilibrium level, y_2. The obverse situation is that in which y_2 does not exist, but *per capita* income is initially below the rate at which population growth attains its maximum. In this case the original fundamental differential equation describes the movement of *per capita* output to the level y^{+}, at which point population growth attains its maximum rate. After y^{+} is attained, the growth of *per capita* output is described as taking place at the constant positive rate, $\alpha - \beta\epsilon$. The intermediate case in which $y^{+} = y_2$ and $\alpha - \beta\epsilon = 0$ seems unlikely. But the resulting movements of the economy would be characterised by movements toward y_2 for any initial output level below $y_2 = y^{+}$. At any higher level the output of agricultural goods is in equilibrium; thus y_2 is stable from below and neutrally stable from above. All solutions $y > y_2$ are neutrally stable equilibrium values for the system, both from above and from below.

So far as the implication of the foregoing analysis for policy is concerned:

Any change in social policy corresponds to an alteration in the parameters of the system. If a society finds itself in a low-level equilibrium trap and the parameter β, representing diminishing returns to additions in the agricultural labour force, remains constant, only two parameters remain to be altered by social policy, namely, α, the rate of technical progress, and ϵ, the maximum net rate of reproduction. If the rate of technical progress can be increased (without the creation of a manufacturing sector) the sign of the expression $\alpha - \beta\epsilon$ may be changed from negative, the trap situation, to positive, in which case there is a steady increase in the output of food *per capita.* An alternative avenue for social policy would be a change in the rate of reproduction. Any improvement in medical technique would be reflected in a reduced force of mortality, so that δ, the previous mortality rate, would be reduced. For the same level of the gross reproduction rate, $\delta + \epsilon$, population would increase at a rate which was more rapid than before; that is, the net rate of reproduction, ϵ, would rise. Thus an improvement in medical technique with no other alterations in the parameters of the model would lead to a decrease in the test criterion, $\alpha - \beta\epsilon$, leading to retardation of the rate of growth. If this criterion is already negative, inspection of the expression for the stationary solution, y_2, reveals that the new stationary level of food output *per capita* is lower than before, so that improvements in medical technique with no other changes lead to immiseration of the surviving members of the population. On the other hand, institutional changes, leading to a restriction of the gross reproduction rate below its physiological maximum, will result in no rise in the stationary level of income; however, they may reduce the critical level of income, y^+, to a level at which it falls below the stationary level, y_2. In this event the test criterion changes from negative to positive and the system enters a phase of steady progress in the level of food output *per capita.* It may be helpful to contrast this model of economic development of a backward economy with the familiar low-level equilibrium trap model of Leibenstein. The only resemblance between the two models is that it is possible to have constant *per capita* output with a growing population. However, in the theory just presented there is at most one equilibrium level of output per head corresponding to each set of parameter values, while in Leibenstein's analysis there are two equilibrium solutions at which *per capita* income is constant, one stable in the small and the other unstable.[1] The implications for social policy are correspondingly different. To escape the low-level equilibrium trap of the present model, changes in the rate of introduction of new techniques or measures of birth control are prescribed; in Leibenstein's analysis the parameters of the model are regarded as essentially fixed and progress is possible only by a massive infusion of capital into the system. The role played by capital accumulation in the development of the agricultural sector

[1] Leibenstein, *op. cit.*

will be discussed in more detail below, after the development of the advanced sector has been described.

3. In this section a rigorous analysis of the fundamental differential equation for a backward economy is presented. Readers who are willing to take the mathematics for granted should go on to the next section. The preceding analysis of the fundamental differential equation revealed that there are essentially two modes of development for an economy which is purely traditional or wholly agricultural—either steady growth in output per man or equilibrium at some level of output per man which remains constant. Any given situation will be described by one of these two cases, depending on the net rate of reproduction, the elasticity of output with respect to increases in the labour supply and the rate of technological progress. In this section we present a detailed proof of the stability of the stationary solution $y_2 = (\alpha + \beta\delta)/\beta\gamma$ where such a solution exists, and the equivalence of the existence of such an equilibrium to the condition $\alpha - \beta\epsilon < 0$. We will also show that if $\alpha - \beta\epsilon > 0$ the long-run steady growth solution will be attained from any positive initial output levels. We begin by solving the fundamental differential equation:

$$\dot{y} = (\alpha + \beta\delta)y - \beta\gamma y^2$$

First, changing variables, let $y = y_2 - 1/u$ define the new variable, u; then the fundamental equation is written:

$$\frac{\dot{u}}{u^2} = (\alpha + \beta\delta)(y_2 - 1/u) - \beta\gamma(y_2 - 1/u)^2$$

Eliminating y_2 and multiplying by u^2 we have:

$$\dot{u} = (\alpha + \beta\delta)u - \beta\gamma$$

which has the solution:

$$u(t) = e^{(\alpha+\beta\delta)t}\left[\frac{1}{y_2 - y(0)} - \frac{\beta\gamma}{\alpha + \beta\delta}\right] + \frac{\beta\gamma}{\alpha + \beta\delta}$$

Substituting this expression into the fundamental differential equation, we obtain:

$$y(t) = y_2 + \frac{1}{e^{(\alpha+\beta\delta)t}\left[\dfrac{\beta\gamma}{\alpha + \beta\delta} + \dfrac{1}{y(0) - y_2}\right] - \dfrac{\beta\gamma}{\alpha + \beta\delta}}$$

which is the general solution for $y(t) < y^{+}$ and for $y(0) \neq y_2$.[1]

Case I. Suppose $y_2 < y^{+}$. Then $0 < (\alpha + \beta\delta)/\beta\gamma < (\epsilon + \delta)/\gamma$, since y_2 is always positive. But since γ is positive this reduces to:

$$\frac{\alpha}{\beta} + \delta < \epsilon + \delta$$

[1] This solution has the form of a " logistic " curve, familiar from many biological and economic applications.

Hence:

$$(\alpha - \beta\epsilon) < 0$$

which is necessary and sufficient for the steady growth path of output corresponding to $y > y^+$ to be characterised by a negative rate of growth. If $y(0) > y^+$ income declines to y^+. If $y_2 < y(0) \leqq y^+$ the general solution reveals that $y(t) - y_2$ diminishes to zero. If $0 < y(0) < y_2$, $y(t) - y_2$ increases to zero. Hence, for any positive initial level of output whatever, y_2 is a stable equilibrium solution. If $y(0) = y_2$, $y(t) = y_2$, since y_2 is an equilibrium of the fundamental differential equation.

Case II. Suppose $y_2 > y^+$. Then $\alpha - \beta\epsilon > 0$ by analogous reasoning and if $y(0) > y^+$ the system grows at the long-run equilibrium rate. If $0 < y(0) < y^+ < y_2$ the general solution has the form:

$$y(t) = y_2 + \frac{1}{e^{(\alpha + \beta\delta)t}\left[\dfrac{1}{y_2} + \dfrac{1}{y(0) - y_2}\right] - \dfrac{1}{y_2}}$$

where $y(0) - y_2$, the initial value of the fraction on the right-hand side, is negative. Since $1/y_2 < 1/(y_2 - y(0)) < 0$, the fraction remains negative and approaches zero from below. But $y(t) = y^+ < y_2$ for t sufficiently large, since $y(t)$ approaches y_2 from below. Hence the equilibrium growth path is stable.

Case III. $y_2 = y^+$. If $0 < y(0) < y^+$, $y(0)$ approaches $y_2 = y^+$ from below. If $y(0) \geqq y_2$, $y(t) = y(0)$ for all t, since $\alpha - \beta\epsilon = 0$ by reasoning analogous to the previous two cases.

Note that the fundamental condition $y^+ \gtreqless y_2$, which determines whether or not a stationary equilibrium exists, is equivalent to the condition $\alpha - \beta\epsilon \lesseqgtr 0$, which determines whether or not such an equilibrium is stable, if it exists. The theoretical analysis which applies to a given case is determined by the three parameters α, β and ϵ, the rate of technical progress, the elasticity of output with respect to land (unity less the elasticity of output with respect to labour) and the net rate of reproduction.

4. In previous sections we presented an analysis of the development of a backward or purely traditional economic system. Depending on the conditions of production and the net reproduction rate, the system is characterised either by a low-level equilibrium, in which output per head is constant and population is growing at less than its physiologically maximum rate, or by a steady growth equilibrium, in which output per head is rising and population is growing at its physiologically maximum rate. Where output per head is constantly rising, an agricultural surplus is generated. Using the notation of the previous discussion, the agricultural surplus, per member of the agricultural labour force, is defined by:

$$y - y^+ = s$$

where s is the agricultural surplus. From the point of view of economic-

demographic development, y^+ is the level of output of food necessary to bring about the maximum rate of increase in population. If agricultural output exceeds this rate, part of the labour force may be freed from the land to produce industrial goods with no diminution in the rate of growth of the total labour force. Let us denote the agricultural population by A and the manufacturing population by M; then the total population is the sum of these two quantities:

$$P = A + M$$

The theory of population growth for a dual economy is essentially the same as that for a backward economy: The net rate of reproduction is the minimum of the physiologically maximum rate and the gross reproduction rate which corresponds to the output of food *per capita* for the total population, less the force of mortality. The function describing the growth of population may be written:

$$\frac{\dot{P}}{P} = \min \begin{cases} \epsilon \\ \gamma y \dfrac{A}{P} - \delta \end{cases}$$

Where $A = P$, the whole labour force is engaged in agricultural production, and the model of population growth reduces to that discussed previously.

In a dual economy labour may be freed from the land at a rate which is just sufficient to absorb the agricultural surplus. Of course, if the growth of manufacturing is not sufficiently rapid some of the excess labour force will remain on the land and part or all of the surplus may be consumed in the form of increased leisure for agricultural workers; this condition will eventuate in the virtual destruction of manufacturing activity or in some arrangement whereby manufactured goods are exported and food is imported. The latter situation is ruled out by our assumption of balance in trade in both classes of commodities. The balance between the expansion of the manufacturing labour force and the production of food is simply described: Total agricultural output is simply output per man employed in agriculture multiplied by the agricultural population. Total food consumption is y^+, multiplied by total population, so that the proportion of the total labour force employed in agriculture is the ratio of the subsistence level of agricultural production to the actual agricultural output per man in the agricultural population:

$$\frac{y^+}{y} = \frac{A}{P}$$

This relationship holds only when an agricultural surplus exists; that is, when there is a positive agricultural surplus rather than a shortage of food, and $y > y^+$. Hence, the relationship governing the distribution of labour between agriculture and industry may be represented by:

$$\frac{A}{P} = \min \begin{cases} 1 \\ y^+/y \end{cases}$$

To complete the model the conditions of production in manufacturing and of capital accumulation in the manufacturing sector must be described. If M is the manufacturing labour force and K is capital stock, then the level of manufacturing output is a function of the quantities of each of these factors employed; but since technical progress in manufacturing may be expected to be quite rapid, the output of manufactured goods for a given bundle of the two factors may be expected to depend not only on the amounts of each of the two factors employed but also on the time at which production takes place. The production function for manufacturing may be written in the form:

$$X = F(K, M, t)$$

where X is manufacturing output. For any given time, it will be assumed that the production function exhibits constant returns to scale; this assumption is equivalent to the assumption that manufacturing output is exhausted by factor payments to labour and to the owners of capital. If, further, the relative share of labour in manufacturing output is constant and all technical change is neutral, the production function may be represented in the Cobb–Douglas form:

$$X = A(t)M^{1-\sigma}K^{\sigma}$$

where $1 - \sigma$ is the relative share of labour and $A(t)$ is some function of time. If the rate of growth is A is constant, say:

$$\frac{\dot{A}}{A} = \lambda$$

then we have:

$$\dot{A} = \lambda A$$

and solving this relation as a differential equation we have:

$$A(t) = e^{\lambda t} A(0)$$

so that the production function takes the form:

$$X = e^{\lambda t}A(0)M^{1-\sigma}K^{\sigma}$$

Dividing X and K by M, and representing output per man and capital per man by x and k, respectively; further, changing the units of X so that $A(0) = 1$, we have the final version of the production function:

$$x = e^{\lambda t} k^{\sigma}$$

Differentiating this function with respect to time and dividing through by output per man, this expression reduces to:

$$\frac{\dot{x}}{x} = \lambda + \sigma\frac{\dot{k}}{k}$$

which may be compared with Kaldor's technical progress function for a

single-sector model.[1] The production function described here implies the existence of a technical progress function which is linear.[2] Whatever the difficulties which exist in expressing output per man as a function of capital per man,[3] it is clear that no distinction can be made between such a production function and a *linear* technical progress function from the point of view of empirical observation. The interpretation of the underlying mechanism is, of course, vastly different in the two cases; but the effects of the two distinct mechanisms cannot be separately identified empirically.

The remaining problem for the theory is this: What determines the rate of capital accumulation? The first approach to this problem is through the fundamental *ex post* identity between the sum of investment and the consumption of manufactured goods, on the one hand, and manufacturing output, on the other. We assume with Kaldor that industrial workers do not save and that property owners do not consume, at least out of their property income.[4] Then, the consumption of manufactured goods, in both the manufacturing and the agricultural sector, is equal to the share of labour in the product of the manufacturing sector. The industrial wage-rate is determined by the usual marginal productivity condition:

$$\frac{\partial M}{\partial X} = (1 - \sigma)x = w$$

where x is output per man, $1 - \sigma$ is the relative share of labour in the total product and w is the industrial wage-rate. The condition that the industrial wage-rate is equal to the marginal product of labour is a necessary condition for the maximisation of profit. While it is not unreasonable to assume that profits are maximised in the advanced sector, there seems to be much less reason for making such an assumption for the agricultural sector. In fact, agricultural workers can be expected to respond to wage differentials between industry and agriculture only if industrial wages are greater than agriculture income, where agricultural income includes both wages and rent. Following the Weber–Fechner law of proportional effect, we can assume that the differential which is necessary to cause movement of agricultural labour into the industrialised sector is roughly proportional to the industrial wage-rate. Let $\mu < 1$ denote the ratio between agricultural

[1] Kaldor, "Capital Accumulation and Economic Growth," mimeographed manuscript, p. 42, equation (14).

[2] This was pointed out to me by F. H. Hahn.

[3] For a discussion of the difficulties of constructing a production function see Joan Robinson, "The Production Function and the Theory of Capital," *Review of Economic Studies*, Vol. 21, 1953–54, pp. 81–106; R. M. Solow, "The Production Function and the Theory of Capital," *ibid.*, Vol. 23, 1955–56, pp. 101–8; Joan Robinson, "Reply," *ibid.*, Vol. 23, 1955–56, p. 247; and recently Joan Robinson, "Some Problems of Definition and Measurement of Capital," *Oxford Economic Papers*, Vol. 11, 1959, pp. 157–66.

[4] N. Kaldor, "Alternative Theories of Distribution," *Review of Economic Studies*, Vol. 23, 1955–56, pp. 83–100; reprinted in *Essays on Value and Distribution* (London: Duckworth, 1960).

income per man and the industrial *wage-rate* per man. Then the total wage bill for the economy is given by the expression:

$$wM + \mu wA = (1 - \sigma)X + qY$$

where wM is the industrial wage bill, μwA is total agricultural income (expressed in manufactured goods), $(1 - \sigma)X$ is total consumption of manufactured goods by workers in both sectors and qY is the value of agricultural output measured in manufactured goods. The variable q is the terms of trade between agriculture and industry. It is assumed that all of agricultural income, whether this income accrues in the form of rent or wages, is consumed. If part of agricultural income, say income from property, were available for investment inside or outside the agricultural sector, the balance relation given above could not hold. It is assumed that all agricultural incomes are consumed, so that investment in the manufacturing sector is financed entirely out of the incomes of property-holders in that sector. The assumption that landholders do not accumulate capital accords with the classical theories of land-rent, especially the theory of Smith and the physiocrats.[1] If agricultural activity is fully rationalised it would be necessary to replace the theory just described by one which is more neoclassical: Agricultural incomes could be divided into property incomes; the agricultural wage-rates (and hence the terms of trade) would be determined by the condition that real wages be equal in the two sectors in the long-run. It is assumed through the analysis that follows that agriculture is traditional in its organisation and that the classical model applies.

Once the share of labour in manufacturing output is distributed to workers in the form of food and consumption goods, and agricultural workers have received the proportion of manufacturing output which must be traded for food, the remainder of manufacturing output is available for capital accumulation, or more properly, for investment. Capital accumulation is defined as investment less depreciation, where depreciation (calculated by the declining-balance method) is a constant fraction of capital stock:

$$\dot{K} = I - \eta K$$

where η is the rate of depreciation, I is investment and $\dot{K}$ is net capital accumulation. Then, by definition, total industrial output is equal to consumption plus investment:

$$X = (1 - \sigma)X + I$$

which implies the following relation between output and capital stock:

$$X = (1 - \sigma)X + \dot{K} + \eta K$$

This relation closes the system and completes the formulation of a theory of development for a dual economy.

[1] See, for example, F. Quesnay, *Tableau Oeconomique* (London: British Economic Association, 1894 (first edition, 1758)).

5. To study the development of a dual economy for the case which is of greatest interest—that in which an agricultural surplus eventually emerges so that development of an advanced sector is possible—we must assume at the outset that $\alpha - \beta\epsilon > 0$, which is necessary and sufficient for the emergence of an agricultural surplus. If no such surplus comes into existence the economy remains in an undeveloped state, producing only food and other products of the traditional sector. The level of production stagnates, that is, remains constant with a rising population, while population increases at less than its maximum rate. The theory of a dual economy under these conditions reduces to the theory of a backward economic system, and nothing need be added to the analysis of the earlier sections. To characterise the development of an advanced sector we will begin by analysing the growth of the manufacturing labour force. This will be followed by a detailed consideration of capital accumulation and growth in the level of output. The analysis will be concluded by a discussion of wages, interest and the terms of trade between industry and agriculture. An industrial labour force comes into being when $y = y^+$, that is, when agricultural output attains the minimum level necessary for population to grow at its maximum rate. From this point forward, population grows at the rate ϵ, the maximum rate of net reproduction. This implies that:

$$P(t) = e^{\epsilon t} P(0)$$

where $t = 0$ is taken as that point of time at which $y = y^+$. From the fact that population is growing at a constant rate and that *per capita* consumption of food is stationary, it is easily seen that food output and population grow at the same rate:

$$\frac{Y}{P} = y^+$$

$$Y = Py^+ = P(0)e^{\epsilon t} y^+$$

Given the agricultural production function, the required rate of growth in the agricultural labour force necessary to maintain the growth of the agritural surplus may be calculated as follows:

$$Y = e^{\alpha t}A^{1-\beta} = P(0)e^{\epsilon t} y^+$$

which implies:

$$A^{1-\beta} = P(0)y^+ e^{[\epsilon - \alpha]t}$$

which may be further simplified:

$$A = [P(0)y^+]^{\frac{1}{1-\beta}} e^{\left[\frac{\epsilon-\alpha}{1-\beta}\right]t}$$

But, recalling the fact that the origin of time is taken as the point at which the maximum rate of population growth begins, we have the following

relation between $P(0)$, total population (entirely employed in agriculture) and y^+, the critical level of agricultural output:

$$y^+ = P(0)^{-\beta}$$

substituting this expression for y^+ into the formulas given above, we obtain the following equivalent expressions for the agricultural labour force:

$$A = P(0)e^{\left[\frac{\epsilon-\alpha}{1-\beta}\right]t} = A(0)e^{\left[\frac{\epsilon-\alpha}{1-\beta}\right]t}$$

where $P(0) = A(0)$. It is easily seen that agricultural population may grow, decline or remain constant, depending solely on the relative magnitude of the two parameters ϵ, the maximum rate of population growth, and α, the rate of technological progress in the agricultural sector.

The manufacturing labour force (or population; no distinction between the two concepts is needed here) is total population less the agricultural population. Hence the growth of the manufacturing labour force is governed by the following expression:

$$M = P(0)\left[e^{\epsilon t} - e^{\left(\frac{\epsilon-\alpha}{1-\beta}\right)t}\right]$$

which is zero at time $t = 0$ and grows at a rate which is always more rapid than the rate of population growth. This is an immediate consequence of the condition equivalent to the existence of a positive agricultural surplus, namely:

$$\alpha - \beta\epsilon > 0$$

which implies:

$$\epsilon - \alpha < \epsilon(1 - \beta)$$

so that:

$$\epsilon > \frac{\epsilon - \alpha}{1 - \beta}$$

so that population is growing more rapidly than the agricultural labour force, hence the manufacturing labour force is growing more rapidly than population, since the rate of growth of population is simply the weighted average of the rates of growth of each of its two components. A second consequence which may be derived from the expression for the size of the manufacturing labour force is that the rate of growth is always declining and approaches, as a limit, the rate of growth of population, ϵ. This result may be deduced from the relative stability of the component $P(0)e^{\epsilon t}$ in the solution for M, the manufacturing labour force.

To study capital accumulation, it is necessary to analyse the relationship between growth of the industrial labour force, investment and output. In addition to the expression for the size of the manufacturing labour force:

$$M = P(0)\left[e^{\epsilon t} - e^{\left(\frac{\epsilon-\alpha}{1-\beta}\right)t}\right]$$

the fundamental relations include the production function:

$$X = e^{\lambda t} K^{\sigma} M^{1-\sigma}$$

and the identity in industrial output:

$$X = (1 - \sigma)X + \dot{K} + \eta K$$

which may be simplified as follows:

$$\sigma X = \dot{K} + \eta K$$

which is, in familiar terms, saving equals investment. Investment is made up of two components—net capital accumulation, $\dot{K}$, and replacement investment, ηK. Using the production function to eliminate X, the level of output in the manufacturing sector, we have:

$$\dot{K} + \eta K = \sigma e^{\lambda t} K^{\sigma} M^{1-\sigma}$$

Using the expression for the size of the manufacturing labour force, we have:

$$\dot{K} = -\eta K + \sigma K^{\sigma} P(0)^{1-\sigma} e^{\lambda t}\left[e^{\epsilon t} - e^{\left(\frac{\epsilon - \alpha}{1-\beta}\right)t}\right]$$

which is the fundamental differential equation for the development of a dual economy. To study capital accumulation it is necessary to solve the fundamental differential equation and to characterise its solutions. In the case of a purely traditional economy it was possible to obtain a solution of the fundamental equation in closed form so that a complete characterisation of the modes of development of the system could be obtained. It is apparently impossible to solve the equation for a dual economy in the same way, so that only long-run tendencies of the system can be characterised.

The first thing to be noted is that there is no stationary situation for any economy in which capital accumulation is possible, that is, for which there is a positive and growing agricultural surplus. Once the economy has begun to grow it must continue to do so. The actual pattern of growth is determined by two fixed initial conditions: First, the size of the total population at the time at which sustained growth begins; secondly, the size of initial capital stock. Of these parameters, only the influence of the initial size of population has any effect on the long-run growth of the economy. The influence of initial capital stock dies out quickly; the greater the rate of depreciation η and the greater the relative share of labour $(1 - \sigma)$, the more rapidly the effects of the initial capital endowment disappear. Secondly, there is no critical level of initial capital endowment below which no sustained growth is possible. Even the smallest initial stock gives rise to sustained growth; that is, the combination of a positive and growing agricultural surplus and a small positive initial capital endowment must give rise to the well-known and much-discussed take-off into sustained capital accumulation and increase in output. If sustained growth can be achieved, the long-run

characteristics of capital accumulation, increases in the level of output and growth of population eventually assume easily recognisable form. For long-run equilibrium growth, capital and output grow at the same rate, even in the presence of neutral technological change. If there is no technical progress in manufacturing, capital, output and population all grow at the maximum rate of growth of population, ϵ. If technical progress takes place, population grows at its maximum rate, ϵ; capital and output grow at a more rapid rate, namely $\lambda/(1 - \sigma) + \epsilon$, where λ is the rate of technical progress and $(1 - \sigma)$ is the share of labour. Growth in the level of manufacturing output is more rapid the greater the rate of growth of the labour force and the more rapid the pace of technological change; growth is less rapid the greater the share of labour in current output or more rapid the greater the saving ratio. The rate $\lambda/(1 - \sigma) + \epsilon$ is a kind of natural rate of growth, Harrod's G_n. The fact that in the long-run the accumulation of capital and the growth of output adjust themselves to the natural rate when substitution in production is permitted is due to Solow,[1] whose analysis of the growth of an advanced economy provides an approximation the model of development for a dual economy which improves as a larger and larger proportion of the total activity of the system shifts from the traditional to the advanced sector. Since capital and output grow at the same rate in long-run equilibrium, the capital–output ratio approaches a fixed value. Whether or not the accelerator is fixed in the short-run, long-run equilibrium growth will be characterised by a constant ratio of capital to output, even in the presence of technological change, factor substitution and an underdeveloped sector of the economy. For an advanced economy characterised by a pattern of development like that of the dual economy we have analysed, the familiar Harrod–Domar growth model is a perfectly valid theory of growth in long-run equilibrium.[2] It must be noted that this result is strictly correct *only* in long-run equilibrium; the validity of the short-run aspect of Harrod's model, with its knife-edge instability under the impact of short-run disequilibrium movements, cannot be assessed here.

We have been able to discuss only the features of the theory of development of a dual economy which characterise long-run equilibrium. One further aspect of the theory which should be noted is the absence of a critical " minimum effort " necessary for take-off into sustained growth. Whatever the initial capital endowment of the advanced sector, sustained growth must eventuate. Secondly, no matter how small the initial endowment, the beginning of growth of manufacturing output is invariably accompanied by a " big push " of activity associated with an extraordinarily high rate of growth of output. But for a dual economy this high initial rate of growth

[1] R. M. Solow, " A Contribution to the Theory of Economic Growth." The result that capital and output grow at the same rate with neutral technological change does not accord with the analysis of Solow, *op. cit.*, p. 86. In correspondence he points out that a minor arithmetical slip in his argument is responsible for the discrepancy.

[2] For the basic references see footnote 3, p. 310 above.

may be viewed as essentially a statistical artifact. Using the production function for the advanced sector, it is possible to derive the relation:

$$\frac{\dot{X}}{X} = \lambda + (1 - \sigma)\frac{\dot{M}}{M} + \sigma\frac{\dot{K}}{K}$$

so that the rate of growth of output in the industrial sector is the rate of technological progress plus a weighted average of the rates of growth of the manufacturing labour force and the rate of growth of capital stock. Since capital stock always grows at some positive rate and the growth of the manufacturing labour force begins at an extremely high rate and declines gradually to the rate of growth of population, the initial rate of growth of manufacturing output must be extremely high, declining gradually and approaching its equilibrium value. The existence of a statistically observable " big push " in manufacturing output is no evidence for the necessity of a massive infusion of capital from outside the system. There may be other reasons for which such an infusion might be desirable;[1] it is clearly unnecessary for development leading to sustained growth.

To complete the discussion we must consider the development of wages, interest and the terms of trade between the advanced and the underdeveloped agricultural sector. First, wages per man must be equal to the share of labour multiplied by output per man. Real wages in manufacturing must therefore rise at the rate $\lambda/(1 - \sigma)$ in long-run steady growth; this may be seen as follows:

$$\frac{\dot{w}}{w} = \frac{\dot{x}}{x} = \left[\frac{\dot{X}}{X} - \frac{\dot{M}}{M}\right] = \left[\frac{\lambda}{1 - \sigma} + \epsilon\right] - \epsilon = \frac{\lambda}{1 - \sigma}$$

since output per man grows at a rate given by the rate of growth of output less the rate of growth of the manufacturing labour force. If there is no technological progress in the advanced sector real wages eventually reach some constant level. If technical progress is possible, real wages rise more rapidly, the more rapid the rate of technological change and the higher the saving ratio. Another way to state the relationship of the growth of real wages and the savings ratio is this: The higher the share of labour in the product of any given period, the more slowly must real wages rise. Unlike wages, the real rate of interest attains a constant level whether or not there is technical progress. Provided that activity in the advanced sector is highly rationalised, that is, predominantly " capitalistic " in its mode of organisation, the rate of interest is determined by the usual marginal productivity condition:

$$\frac{\partial X}{\partial K} = \sigma\frac{X}{K} = r$$

[1] A persuasive case for capital-intensive investments is made by H. Leibenstein and W. Galenson, " Investment Criteria, Productivity and Economic Development," *Quarterly Journal of Economics*, Vol. 70, 1955, pp. 343–70; see also: H. Leibenstein, *Economic Backwardness and Economic Growth*, Chapter 15.

which is necessary for maximisation of profit in the advanced sector. But the ratio of capital to output is constant in long-run sustained growth; where C is the capital–output ratio of Harrod, this yields the familiar:

$$r = \frac{\sigma}{C}$$

where σ is the saving ratio. The rate of interest is equal to Harrod's warranted rate, which is essentially the gross rate of capital accumulation or I/K, in the notation of the last section. The rate of growth of output (and the rate of net capital accumulation) is less than the warranted rate by the amount of depreciation on capital stock. This implies that:

$$r = \frac{\sigma}{C} \quad \left[\frac{\lambda}{1-\sigma} + \epsilon\right] + \eta = G_n + \eta$$

where G_n is Harrod's notation for the natural rate of growth. The rate of interest rises with an increase in the rate of technical progress, with an increase in the rate of population growth or in the rate of depreciation on capital, and with a rise in the saving ratio. Another way to state the relation between the rate of interest (the rate of gross capital accumulation) and the rate of depreciation on capital is this: Interest increases with a decline in the durability of capital equipment. Another way to derive the basic relationship between interest and the rate of growth on capital stock is to note that all property income is " interest " in the sense in which interest is used here, so that the total volume of interest payments is equal to gross investment:

$$rK = \dot{K} + \eta K$$

which implies:

$$r = \frac{\dot{K}}{K} + \eta$$
$$= G_n + \eta$$

as before. From these relationships it is possible to compute the long-run sustained growth capital–output ratio, Harrod's C. The capital–output ratio is given by:

$$C = \frac{\sigma(1-\sigma)}{\lambda + (1-\sigma)(\epsilon + \eta)}$$

which increases with the saving ratio and diminishes with the rate of growth of population, the rate of technical progress and the rate of depreciation on capital goods. Turning finally to the terms of trade between agriculture and industry, we have:

$$wM + \mu wA = (1-\sigma)X + qY$$

as the balance relation between industry and agriculture. This reduces to the simpler expression:

$$\mu wA = qY$$

This may be used to derive the expression:

$$\mu w P(0) e^{\left(\frac{\epsilon-\alpha}{1-\beta}\right)t} = q e^{\epsilon t} y^{+}$$

hence:

$$q = w \frac{\mu}{y^{+}} e^{\left[\left(\frac{\epsilon-\alpha}{1-\beta}\right) - \epsilon\right]t}$$

from which the following expression:

$$\frac{\dot{q}}{q} = \left[\frac{\epsilon-\alpha}{1-\beta} - \epsilon\right] + \frac{\dot{w}}{w}$$

is derived by differentiating both sides with respect to time and then by dividing through by q, where q is the terms of trade, that is, the price of agricultural goods in terms of manufactured goods. The first two terms of this expression are negative by the condition that $\alpha - \beta\epsilon > 0$, which is necessary and sufficient for the emergence of an agricultural surplus. The third term, the rate of growth of real wages, is positive and equal to $\lambda/(1-\sigma)$. If technical progress in the manufacturing sector declines to zero the terms of trade for the agricultural sector must deteriorate. The more rapid the rate of technical progress in manufacturing, the less rapidly the terms of trade for agriculture deteriorate; if technical progress is sufficiently rapid the terms of trade may even improve for agriculture. Not unexpectedly, precisely the opposite holds for changes in the terms of trade with respect to changes in the rate of technical progress in agriculture. The more rapid the rate of technical progress, the more rapidly the terms of trade deteriorate. A second consequence of the fundamental balance relation is that the more rapidly population increases, the less rapidly the terms of trade of agriculture deteriorate. Finally, the effects of diminishing returns in agriculture depend on the relative magnitude of technical progress in agriculture and the rate of growth of population. If $\epsilon > \alpha$ the terms of trade improve if returns diminish more rapidly than before; if $\epsilon = \alpha$ the terms of trade remain unaltered by a change in the rate at which returns to additions in the agricultural labour force diminish; finally, if $\epsilon < \alpha$ the terms of trade deteriorate if returns to scale in agriculture diminish more rapidly.

6. In this section a rigorous analysis of the fundamental differential equation of the theory of development of a dual economy is given. This analysis is necessary to provide a detailed derivation of the conclusions of the previous section; as in Section 3 above, readers who trust the mathematics should go on to the following section. The fundamental differential equation is given by:

$$\dot{K} = -\eta K + \sigma K^{\sigma} P(0)^{1-\sigma} e^{\lambda t} \left[e^{\epsilon t} - e^{\left(\frac{\epsilon-\alpha}{1-\beta}\right)t}\right]^{1-\sigma}$$

where K is the size of capital stock, P is the size of the total population, and the various parameters are derived from the expression for growth of the manufacturing labour force, the production function for the advanced sector

and the identity in manufacturing output and its allocation to consumption and investment. To solve this equation, we begin by transforming variables so that:

$$K = U^{\frac{1}{1-\sigma}}$$

hence,

$$\dot{K} = \frac{1}{1-\sigma} U^{\frac{\sigma}{1-\sigma}} \dot{U}$$

The fundamental equation may be rewritten in the form:

$$\begin{aligned}\dot{U} &= -(1-\sigma)\eta U + \sigma(1-\sigma)P(0)^{1-\sigma} e^{\lambda t}\left[e^{\epsilon t} - e^{\left(\frac{\epsilon-\alpha}{1-\beta}\right)t}\right]^{1-\sigma} \\ &= -(1-\sigma)\eta U + q(t)\end{aligned}$$

which is linear in U and may be integrated as follows:

$$U(t) = e^{-(1-\sigma)\eta t}\, U(0) + e^{-(1-\sigma)\eta t} \int_0^t e^{(1-\sigma)\eta t_1}\, q(t_1)\,dt_1$$

Now, consider the integral:

$$\int_0 e^{(1-\sigma)\eta t_1}\, q(t_1)\,dt_1$$

$$= \sigma(1-\sigma)P(0)^{1-\sigma} \int_0^t e^{[(1-\sigma)(\epsilon+\eta)+\lambda]t_1}\left[1 - e^{\frac{-(\alpha-\beta\epsilon)}{1-\beta}t_1}\right]^{1-\sigma} dt_1$$

$$= \sigma(1-\sigma)P(0)^{1-\sigma} \int_0^t e^{pt_1}\,[1 - e^{-qt_1}]^r\, dt_1$$

where p, q and r are defined as in the second line above and are used to simplify notation for the computations that follow. It is apparently impossible to evaluate this integral in closed form except for special cases (such as $r = 1$, which is ruled out for the problem at hand). It is necessary to expand $[1 - e^{-qt_1}]^r$ in Maclaurin series as follows:

$$\begin{aligned}[1 - e^{qt_1}]^r = 1 - re^{-qt_1} + \frac{r(r-1)}{2!}e^{-2qt_1} - \ldots \\ + (-1)^n \frac{r(r-1)\ldots(r-n+1)}{n!}e^{-nqt_1} + \ldots\end{aligned}$$

Hence, the integral is written:

$$\int_0^t \left[e^{pt} - re^{(p-q)t_1} + \frac{r(r-1)}{2!}e^{(p-2q)t_1} - \ldots + (-1)^n \frac{r(r-1)\ldots(r-n+1)}{n!}e^{(p-nq)t_1} + \ldots\right]$$

Integrating term by term we obtain:

$$\int_0^t e^{pt_1}\,[1 - e^{-qt_1}]^r dt_1$$

$$=\frac{1}{p}[e^{pt}-1]-\frac{r}{p-q}[e^{(p-q)t}-1]+\frac{r(r-1)}{2!(p-2q)}[e^{(p-2q)t_1}-1]-\ldots$$
$$+(-1)^n\frac{r(r-1)\ldots(r-n+1)}{n!(p-nq)}[e^{(p-nq)t}-1]+\ldots$$

Hence:

$$U(t)=e^{-(1-\sigma)\eta t}\,U(0)+\frac{\sigma(1-\sigma)P(0)^{1-\sigma}}{(1-\sigma)(\epsilon+\eta)+\lambda}e^{[\lambda+\epsilon(1-\sigma)]t}+R(t)$$

where $R(t)$ is a remainder, defined as the sum of the appropriate infinite series. Each term of $R(t)$ grows at a rate which is strictly less that the rate $\lambda+\epsilon(1-\sigma)$, which is the " dominant " term in the expression for $U(t)$. If $p>q$, $R(t)\rightarrow 0$ as t increases without limit; otherwise $R(t)$ grows over time, diminishing relative to the dominant term. Noting that the dominant term, which grows at a positive rate, and the remainder depend only on the initial value of population, and further that $U(0)=K(0)^{1-\sigma}$, it is clear that the effects of the initial endowment of capital die out, absolutely, and relative to the dominant term. To compute the long-run equilibrium value of the rate of growth for capital stock we note first that:

$$\frac{\dot{U}}{U}\rightarrow[\lambda+\epsilon(1-\sigma)]\text{ as }t\rightarrow\infty$$

But also, using the expressions for U and $\dot{U}$ derived previously we have:

$$\frac{\dot{K}}{K}=\frac{1}{(1-\sigma)}\frac{U^{\frac{\sigma}{1-\sigma}}\dot{U}}{U^{\frac{1}{1-\sigma}}}=\frac{1}{(1-\sigma)}\frac{\dot{U}}{U}$$

so that:

$$\frac{\dot{K}}{K}\rightarrow\frac{\lambda}{(1-\sigma)}+\epsilon\text{ as }t\rightarrow\infty$$

Secondly, using the expression:

$$\frac{\dot{X}}{X}=\lambda+(1-\sigma)\frac{\dot{M}}{M}+\sigma\frac{\dot{K}}{K}$$

we have the result that:

$$\frac{\dot{X}}{X}\rightarrow\lambda+(1-\sigma)\epsilon+\sigma\left[\frac{\lambda}{(1-\sigma)}+\epsilon\right]=\frac{\lambda}{(1-\sigma)}+\epsilon\text{ as }t\rightarrow\infty$$ [1]

so that capital and output grow at the same rate in long-run equilibrium. To compute the long-run equilibrium capital–output ratio, we can compute the ratio of the dominant components in the expressions for each of the terms, capital and output.

[1] This expression may be compared with an essentially equivalent expression derived by N. Kaldor, " A Model of Economic Growth," *loc. cit.*, p. 615. Neo-classical, Kaldor and dual economy models of growth all tend, in the limit, to the equilibrium growth path described by the Harrod–Domar model.

First, we may utilise the production function to derive:

$$\begin{aligned} X(t) &= e^{\lambda t} P(0)^{1-\sigma} \left[e^{\epsilon t} - e^{\left(\frac{\epsilon - \alpha}{1-\beta}\right) t} \right]^{1-\sigma} K(t)^{\sigma} \\ &= e^{\lambda t} P(0)^{1-\sigma} \left[e^{\epsilon t} - e^{\left(\frac{\epsilon - \alpha}{1-\beta}\right) t} \right]^{1-\sigma} U(t)^{\frac{\sigma}{1-\sigma}} \end{aligned}$$

and we also have:

$$K(t) = U(t)^{\frac{1}{1-\sigma}}$$

so that:

$$\begin{aligned} C(t) &= \frac{K(t)}{X(t)} = \frac{U(t)}{e^{\lambda t} P(0)^{1-\sigma} \left[e^{\epsilon t} - e^{\left(\frac{\epsilon - \alpha}{1-\beta}\right) t} \right]^{1-\sigma}} \\ &\to \frac{\dfrac{\sigma(1-\sigma)P(0)^{1-\sigma}}{(1-\sigma)(\epsilon+\eta)+\lambda} e^{[\lambda \times \epsilon(1-\sigma)]t}}{P(0)^{1-\sigma} e^{[\lambda - \epsilon(1-\sigma)]t}} \\ &= \frac{\sigma(1-\sigma)}{(1-\sigma)(\epsilon+\eta)+\lambda} \text{ as } t \to \infty \end{aligned}$$

This completes the analysis of the fundamental differential equation.

7. The development of a dual economy has been discussed under the assumption that if an agricultural surplus comes into existence it will persist. If such a surplus is already in existence, and a change in the parameters of the system, particularly in the net rate of reproduction, results in a diminution of the agricultural surplus, implying its eventual disappearance, the course of development of the economy is considerably different. Eventually manufacturing activity is brought to a halt, capital is allowed to depreciate without replacement and the situation previously described as a low-level equilibrium trap is approached as a limit. If a low-level equilibrium trap exists, it is stable for any initial level of agricultural and industrial output, and for any initial stock of capital. To trace the decline of the economy to its trap level of output, note first that from the point at which the agricultural surplus begins to diminish, the agricultural labour force grows at a rate which is more rapid than the rate of growth of population and the manufacturing force declines absolutely, eventually becoming zero at some finite time, which is easily computed. At the point at which all labour has returned to the land, manufacturing output drops to zero and capital is decumulated at the rate given by the rate of depreciation; in the limit it disappears entirely. From the point at which industrial output drops to zero, the theory of a traditional economic system governs the further decline of the system to its trap level. Population growth is reduced from its maximum rate. Food output *per capita* declines to a stationary level.

The critical condition, which marks the dividing line between economies caught in the low-level equilibrium trap and economies capable of sustained growth, is simply that for sustained growth an agricultural surplus must

come into existence and must persist.[1] Formally, the condition which is necessary and sufficient for sustained growth of output in both the manufacturing and agricultural sectors is $\alpha - \beta\epsilon > 0$, where α is the rate of technical progress, ϵ is the maximum rate of population growth and $1 - \beta$ is the elasticity of output in the agricultural sector with respect to an increase in the agricultural labour force. The characteristics of an economy which moves toward a low-level equilibrium trap are completely described by the theory of a traditional or backward economy. The characteristics of an economy which experiences steady growth depend not only on the existence of an agricultural surplus but also on technical conditions in the advanced sector. The more rapid the rate of technical change, the higher the saving ratio, and the more rapid the rate of growth of population, the more rapid is the pace of growth in the advanced sector. Eventually, the economy is dominated by the development of the advanced sector and becomes more and more like the advanced economic systems described by the familiar Harrod–Domar theory of economic growth.

Of course, the theory of a dual economy is far from the whole story concerning the growth of an advanced economic system. From the outset the possibility of capital accumulation in agriculture was ruled out. While this assumption holds remarkably well for Asian agriculture, including the highly productive agricultural sector of the Japanese economy, in the " newly settled " lands of the United States, Canada, Argentina, Australia and New Zealand agriculture has experienced rapid increases in labour productivity due to infusions of capital. We have already noted that there is no " minimum critical effort " of investment in dual economies in which substitution between capital and labour in manufacturing is possible, and there is no capital accumulation in agriculture; however, one of the critical parameters of the expression dividing economies into developing and stationary systems is the rate of technological progress in agriculture. If technical progress can be accelerated by the accumulation of capital in agriculture the balance between food shortage and agricultural surplus may be tipped in favour of surplus. It must be remarked that this possibility presumes not only the availability of resources for investment in agriculture but also drastic changes in the social organisation of the " traditional " or backward sector. The infusion of capital must be accompanied by the infusion of the spirit of capitalism;[2] the traditional way of life must be replaced by a way of life in which production in both agriculture and industry is fully rationalised.

DALE W. JORGENSON

University of California, Berkeley.

[1] This condition is given its due weight by Kaldor in his paper " Characteristics of Economic Development " read to the International Conference on Underdeveloped Areas in October 1954, now printed in *Essays on Economic Stability and Growth* (London: Duckworth, 1960), pp. 233–42. See especially the discussion of this problem on p. 240.

[2] The rationalisation of production in the traditional sector is, obviously, one of the great " costs " of economic advancement in underdeveloped areas. The view that the cost is too great to pay is not without its adherents; see especially the work of Boeke, cited in footnote 3, p. 310.